Stability in S. S.
&
Steady-State Errors

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6.5 Stability in state space :

→ The eigenvalues of the matrix A are identical to the system's poles.

⇒ So, we can determine the stability of a system represented in S.S. by finding the eigenvalues of the system matrix A.

Problem : Given the system

$$\dot{x} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]\, x$$

find out how many poles are in the left half in the right half-plane, and on the jw-axi

$$(sI - A) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} = \begin{bmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix}$$

$$\det(sI - A) = s[(s-8)(s+2) + 5] + 3[-2(s+2) + 10] - 1[-10$$

$$s[s^2 - 6s - 11] + 3[-2s + 6] - [-10s + 70]$$

$$= s^3 - 6s^2 - 11s - 6s + 18 + 10s - 70$$

$$= s^3 - 6s^2 - 7s - 52$$

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$s^3 - 6s^2 - 7s - 52 = 0$

s^3	1	-7
s^2	-6	-52 -26
s^1	$b_1 = \frac{-47}{3}$	$b_2 = 0$
s^0	$c_1 = -26$	

$$b_1 = \frac{42 + 52}{-6} = \frac{94}{6} = 15.$$

Since there is one sign change in the first column, the system has one right half plane

$\therefore$ System is unstable.

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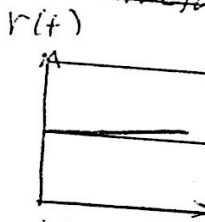
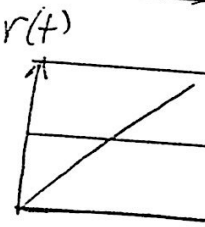
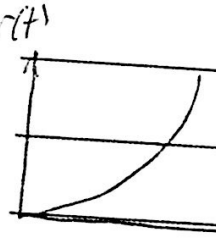
Steady-State Errors: ($e(\infty)$, e_{ss})

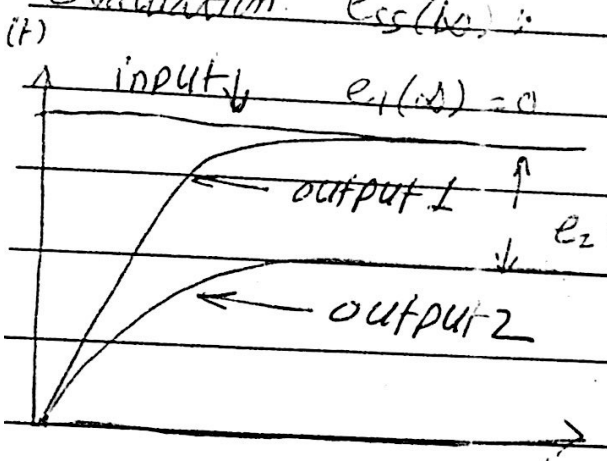
Definition and Test Inputs:

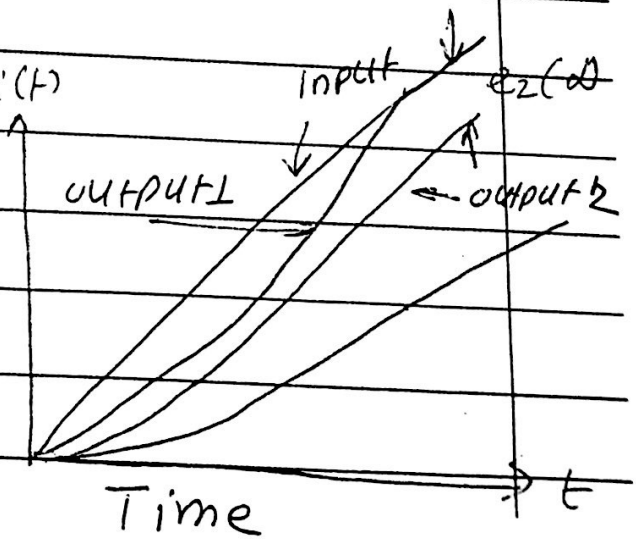
Steady-state error is the difference between the input and the output for a prescribed test input as $t \to \infty$.

Test inputs used for e_{ss} analysis:

Waveform	Name	Physical Meaning	Time function	LT.
	Step	Constant position	1	$1/s$
	Ramp	Constant Velocity	t	$1/s^2$
	Parabolic	Constant acceleration	$\frac{1}{2}t^2$	$1/s^3$

Evaluation e_{ss}(lec.):





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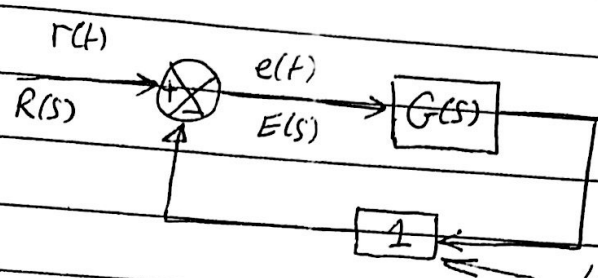
→ Steady State Error for Unity feedback systems:



We assume that CL system is stable!

Unity feedback

→ Suppose that we want output C(t) to track r(t).

→ Error $e(t) = r(t) - C(t)$

→ Steady-state error

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

↑ Final value theorem

② Steady state error for step r(t)

→ For unity feedback system above:

$$E(s) = R(s) - C(s)$$

$$C(s) = E(s)\,G(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

$$e_{ss} = e(\infty) = \lim_{s \to 0} s \cdot \frac{R(s)}{1 + G(s)}$$

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Step input: $R(s) = 1/s$

$$e_{step}(\infty) = \lim_{s \to 0} \frac{s\,(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

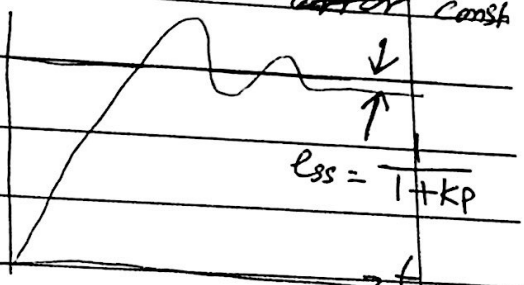
$$k_p = \lim_{s \to 0} G(s) = G(0)$$ (dc gain) or constant position const

$$e_{ss} = \frac{1}{1 + k_p}$$

Note:

→ In order to have zero e_{ss}

$$\lim_{s \to 0} G(s) = \infty$$



Ramp Input: $R(s) = 1/s^2$

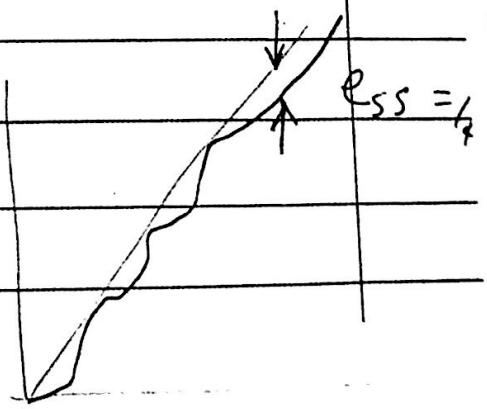
$$e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1 + G(s)} = \frac{s\,(1/s^2)}{1 + G(s)} = \frac{1}{s + sG(s)}$$

$$e_{ss} = \frac{1}{\lim_{s \to 0} s\,G(s)} = \frac{1}{K_v}$$ ← velocity constant

$$K_v = \lim_{s \to 0} s\,G(s)$$

note: Inorder to have zero ess we mus

have: $\lim_{s \to 0} s\,G(s) = \infty$



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Ramp input $R(s) = 1/s^2$

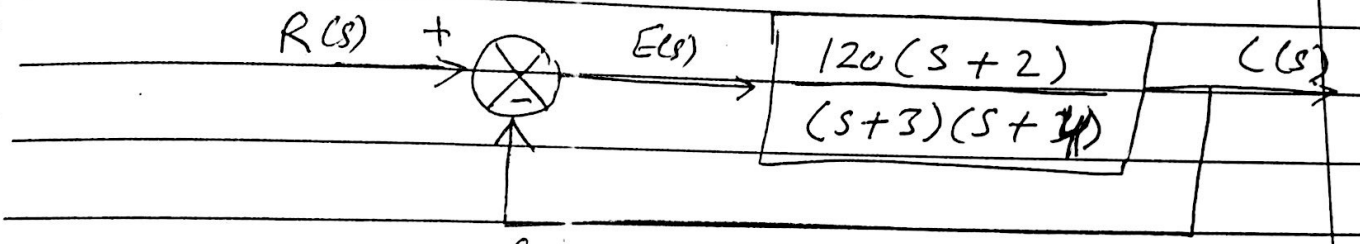
$$e_{ss} = \lim_{s\to 0} \frac{s(1/s^2)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

$$e_{ss} = \frac{1}{K_a}$$ where $K_a = \lim_{s\to 0} s^2 G(s)$ acceleration error const.

Note:
In order to have zero e_{ss} for parabolic. $\lim_{s\to 0} s^2 G(s) = \infty$

→ K_p, K_v, K_a have the ability to reduce steady-state error.

Ex 7.1: Find e_{ss} for inputs of $5u(t)$, $5tu(t)$, $5t^2u(t)$ to the system shown below:



→ Verify the cl system is stable!

Chara. $P(s) = 1 + G(s) = 0$

eq. $$1 + \frac{120(s+2)}{(s+3)(s+4)} = 0$$

$$(s^2 + 7s + 12) + 120s + 240 = 0$$

$$s^2 + 127s + 252 = 0$$

$$s_{1,2} = \frac{-127 \pm \sqrt{(127)^2 - 4(1)(252)}}{2(1)} = -124.9, -2.016$$

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→ for input $r(t) = 5u(t) = \frac{5}{1}s^{-1}$

$$e_{ss} = \lim_{s\to 0} \frac{5}{1 + \lim_{s\to 0} G(s)} = \frac{5}{1+20} = \frac{5}{2}$$

$$K_p = \lim_{s\to 0} \frac{120(s+2)}{(s+3)(s+4)} = 20$$

→ for input $r(t) = 5tu(t) \xrightarrow{LT} \frac{5}{s^2}$

$$e_{ss} = \lim_{s\to 0} \frac{5}{sG(s)} = \frac{5}{0} = \infty$$

$$K_v = \lim_{s\to 0} sG(s) = s \cdot \frac{120(s+2)}{(s+3)(s+4)} = 0$$

$$e_{ss} = \infty$$

→ for input $r(t) = 5t^2u(t) = \frac{5}{s^3}$

$$e_{ss} = \frac{5}{K_a} = \frac{5}{\lim_{s\to 0} s^2 G(s)} = \frac{5}{0} = \infty$$

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Static Error Constants and System Type:

The steady-state error performance specifications are called static error constants their names are;

Position constant, K_p, where

$$K_p = \lim_{s \to 0} G(s)$$

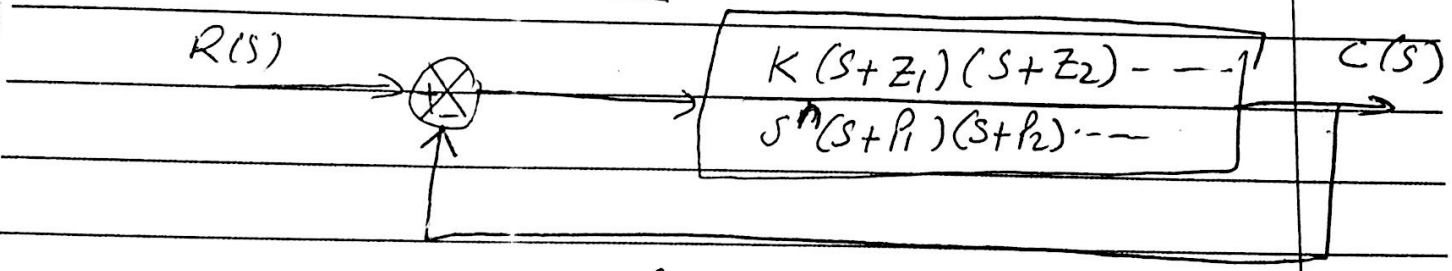
Velocity constant, K_v, where

$$K_v = \lim_{s \to 0} sG(s)$$

acceleration constant, K_a, where

$$K_a = \lim_{s \to 0} s^2 G(s)$$

System Type:



→ We define system type to be the val of n. in the denominator, the number of pure integrations in the forward path

∴ a system ⇒ n=0 is a Type 0

n=1 is a Type 1

n=2 is a Type 2

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→ The values of static error constants depend upon what is the system type to!

→ Relationships between input, system ty. static error constants, and ess:

input	ess formula	Type 0 static error Constant	Type 0 Error	Type 1 static error constant	Type 1 Error	Type static error Constant	
step $u(t)$	$\frac{1}{1+K_p}$	K_p = Constant	$\frac{1}{1+K_p}$	∞	0	∞	
ramp $t\,u(t)$	$\frac{1}{K_v}$	0	∞	K_v = Constant	$\frac{1}{K_v}$	∞	c
parabolic $\frac{t^2}{2}u(t)$	$1/K_a$	0	∞	0	∞	K_a = Constant	

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